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The mixing length parameter α

V.M. Canuto

NASA, Goddard Space Flight Center, Institute for Space Studies, 2880 Broadway, New York, NY 10025, USA

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Abstract. The standard Mixing Length Theory, MLT, treats turbulent eddies as if they were isotropic, while the largest eddies that carry most of the flux are highly anisotropic. Recently, an anisotropic MLT was constructed and the relevant equations derived. In this paper, we show that these new equations can actually be cast in a form that is *formally identical* to that of the standard isotropic MLT, *provided* the mixing length parameter, derived from stellar structure calculations, is interpreted as an intermediate, auxiliary function $\alpha(x)$, where x, the degree of anisotropy, is given as a function of the thermodynamic variables of the problem. The relation between $\alpha(x)$ and the physically relevant $\alpha(l=\alpha H_p)$ is also given. Once the value α is deduced, is found to be a function of the local thermodynamic quantities, as expected.

Key words: convection – turbulence – stars: structure of – Sun: structure of

1. Introduction

The Mixing Length Theory, MLT, suffers from two well known problems. The *first* problem is that the mixing length l is not given by the theory itself and is thus usually written as

$$l = \alpha H_p\,, \tag{1}$$

where H_p is the pressure scale height and α a free parameter. Stellar structure calculations yield a global value of α throughout the entire convective region of around

$$\alpha = 1 - 2 \tag{2}$$

rather than a variable local value as it might be expected. This problem has motivated proposals for more complete mixing length models none of which is however immune from contradictions (for a recent review, see Renzini, 1987). Earlier attempts to formulate an expression for l from first principles (Roxburgh, 1978) have also been criticized (Baker and Kuhfuss, 1987). Thus, it seems fair to say that a consistent expression for the mixing length l is not yet available and that a global value of α is difficult to understand physically.

A *second* problem, which however has received considerable less attention, is that the MLT assumes that all length scales entering the problem are equal to a mixing length l, while a realistic model ought to account for the fact that the large eddies that carry most of the flux are in general anisotropic, i.e., their characteristic width $l_\perp$ and height l_z need not be identical.

Recently, the *second* problem was dealt with (Canuto, 1989; cited as AMLT) and an anisotropic mixing length model proposed which yields the expressions for the convective flux and velocities for an arbitrary degree of eddies anisotropy x, where

$$x = l_z^2/l_\perp^2\,. \tag{3}$$

In this paper, we shall point out a *new feature*, namely that the AMLT equations can actually be formally written as the standard MLT expressions provided the mixing length parameter derived from stellar structure calculations is interpreted as $\alpha(x)$, as shown below. An expression for the physically meaningful α is then derived and shown to depend on the local thermodynamic quantities.

2. The expression for α

Once an expression for the turbluent convective flux F_c is adopted, one derives an expression relating $\nabla - \nabla_{ad}$, $\nabla_r - \nabla_{ad}$ and the local variables represented by a dimensionless variable U, see Eq. (6b) (Cox and Giuli, 1968, use the variable A, where $2UA = 1$). This relation is a cubic equation for the auxiliary function ξ

$$\xi^2 = U^2 + \nabla - \nabla_{ad} \tag{4}$$

or equivalently, the convective efficiency $2\Gamma = \xi/U - 1$.

In the AMLT paper it was shown that even in the presence of an arbitrary degree of anisotropy, the relation yielding $\nabla - \nabla_{ad}$ is *still* a cubic equation *formally* identical to the cubic relation of the standard MLT, i.e.,

$$[\xi(x) - U(x)]^3 + \xi^2(x)\,U(x) - U^3(x) - U(x)\,(\nabla_r - \nabla_{ad}) = 0\,, \tag{5}$$

where the parameter U has become a function $U(x)$ given by

$$U(x) = (8^{1/2}/9)\,x^{-1/2}\,(1+x)^{3/2}\,U\,, \tag{6a}$$

where the MLT parameter U is defined as (Kippenhahn, 1963)

$$U = 24\,\sigma T^3\,(c_p \varrho^2 \kappa l^2)^{-1}\,(2H_p/g)^{1/2}\,. \tag{6b}$$

The fact that the basic cubic Eq. (5) has retained its formal MLT structure even in the presence of an arbitrary degree of anisotropy x, suggests an interesting interpretation. Combining (6), we can write

$$U(x) = (l/l(x))^2\,U\,, \tag{7}$$

where (for $x > 1$)

$$l(x) = (3/8^{1/4})\,(1+x)^{-1/2}\,l\,. \tag{8}$$

It is easy to see that the introduction of the quantity $l(x)$ formally transforms the anisotropic problem into an isotropic one. Since by definition $l \equiv l_z$, substituting l from (8) into (3) and considering large x's, we obtain $l_\perp = l(x) \equiv l_\perp(x)$, i.e., we formally recover an isotropic eddy.

It follows that one need not actually solve the full AMLT equations. Rather, one can solve the cubic equation of the standard MLT model provided that the parameter determined from stellar structure calculations is interpreted as

$$\alpha(x)\,, \tag{9}$$

since in the expression for $U(x)$ one must write

$$l(x) = \alpha(x)\, H_p \tag{10}$$

which, together with (8) then yields

$$\alpha(x) = (3/8^{1/4})\,(1+x)^{-1/2}\,\alpha\,. \tag{11}$$

Which of the two α's is the one physically relevant? Since α is related to the only physical length off the problem, i.e., $l \equiv l_z = \alpha H_p$, it seems that α rather than $\alpha(x)$ is the physically meaningful quantity since it measures the degree of stretching of an eddy in the z direction (in units of H_p), while the quantity $\alpha(x)$ is a convenient mathematical parameter referring to an (idealized) isotropic situation. Since stellar structure calculations indicate that

$$\alpha(x) = 1 - 2\,, \tag{12}$$

we shall write

$$\alpha = \alpha_0 (1+x)^{1/2}\,, \quad \alpha_0 = 8^{1/4}\,\alpha(x)/3 \approx 1 \tag{13}$$

which yields an expression for α which depends on the local variables once a model for x is given.

3. Two models for x

As in the AMLT paper, we suggest that the degree of anisotropy x be determined by maximizing the rate $n_s(k)$ at which energy is injected into the system so as to keep turbulence from decaying. Explicitly, we shall write

$$\frac{d}{dx}\, n_s(x, \nu_*, \chi_*) = 0\,, \tag{14}$$

where the rate n_s is given by the dispersion relation

$$(n_s + \nu_* k^2)\,(n_s + \chi_* k^2) = g\,\alpha\beta x\,(1+x)^{-1} \tag{15}$$

with $\alpha\beta = H_p^{-1}(\nabla - \nabla_{ad})$. In (15), ν_* and χ_* represent an effective viscosity and an effective thermometric conductivity. Furthermore, the maximization (14) is carried out at the lower wavenumber $k = k_0 = (\pi/d)\,(1+x)^{1/2}$.

We shall now discuss two models which differ in the treatment of ν_* and χ_*.

3.1. First model: weak turbulence

In the first model, it is assumed that

$$\nu_* = \nu\,, \quad \chi_* = \chi\,, \tag{16}$$

where ν and χ are the molecular values. Since in the cases of interest in stellar structure studies the viscosity ν is considerably smaller that χ, ν is usually neglected. This is the approximation adopted in the standard MLT as well as in the AMLT. In this case, the result of (14) is given by Eq. (40) of the AMLT paper which we shall rewrite as

$$4(x-1)\,(x+1)^3 = S\,, \tag{17a}$$

$$S = (81/8)\;U^{-2}(\nabla - \nabla_{ad})\,. \tag{17b}$$

Substitute now x from (17a) into (13). Considering that

$$U = \alpha^{-2}\,U_p\,, \tag{18}$$

where

$$U_p \equiv 24\,\sigma T^3\,(c_p\,\varrho^2\,\kappa H_p^2)^{-1}\,(2\,H_p/g)^{1/2} \tag{19a}$$

$$S_p = (81/8)\;U_p^{-2}\,(\nabla - \nabla_{ad}) \tag{19b}$$

we finally obtain

$$\alpha = \alpha_0\,[1 + (1 + \alpha_0^4\, S_p/4)^{1/2}]^{1/2}\,. \tag{20}$$

Values of α are listed in Table 1. For large values of S_p, Eq. (20) gives, with $\alpha_* = 2^{-1/2}\,\alpha_0^2$

$$\alpha = \alpha_*\, S_p^{1/4}\,. \tag{21}$$

Table 1. The parameter α as a function of S_p for weak (W) and strong (S) turbulence. In the first case, see Eq. (20). In the second case, Eq. (29) gives an excellent approximation

S_p	α(S)	α(W)
10^3	0.934	1.381
$3.16\ 10^3$	0.963	1.673
10^4	1.009	2.294
$3.16\ 10^4$	1.063	3.016
10^5	1.127	3.992
$3.16\ 10^5$	1.202	5.268
10^6	1.287	7.050
$3.16\ 10^6$	1.382	9.386
10^7	1.488	12.51
$3.16\ 10^7$	1.604	16.67
10^8	1.729	22.23
$3.16\ 10^8$	1.864	29.64
10^9	2.00	39.52
$3.16\ 10^9$	2.164	52.69
10^{10}	2.329	70.27
$3.16\ 10^{10}$	2.507	93.68
10^{11}	2.695	124.95
$3.16\ 10^{11}$	2.897	166.59
10^{12}	3.113	222.2

3.2. Second model: strong turbulence

Since the most distinguishing feature of turbulence is its ability to greatly enhance all transport properties over their molecular values, it is clear that the approximation (16) cannot hold true in a fully turbulent regime. Its validity is limited to a regime near the inception of turbulence, i.e., not far from the laminar regime. However, since we need to describe a fully developed turbulent regime, as the one that attains in the convective regions of stars where the equivalent of the Reynolds number is extremely large, a better representation for ν_* and χ_* is required. As customary in theory of turbulence, we shall write

$$\nu_* = \nu + \nu_t \quad \chi_* = \chi + \chi_t \,, \tag{22}$$

where ν_t and χ_t represent the renormalization of the molecular values due to the presence of turbulence. Ordinarily, $\nu_t \gg \nu$ and $\chi_t \gg \chi$. The expressions for the turbulent viscosities and conductivities can be obtained once the energy spectral function $E(k)$ characterizing the turbulent regime is known. In fact (Canuto et al., 1987)

$$\nu + \nu_t = (\nu^2 + \gamma \int k^{-2} E(k)\, dk)^{1/2} \,, \tag{23}$$

where the integration limits are k_0 which depends on x, and infinity. The numerical coefficient γ is related to the Kolmogoroff constant Ko by the relation $\gamma = (2/3\,\mathrm{Ko})^3$. An analogous expression holds for the turbulent conductivity with $\gamma \to \xi = 1/3$. Taking $E(k)$ to be of the Kolmogoroff type $E(k) = \mathrm{Ko}\, \varepsilon^{2/3} k^{-5/3}$, the integration can easily be performed. The important fact is that ε, the energy per sec per gram fed to the largest eddies by the external source, is related to the convective flux F_c since one can show that (Canuto et al., 1987)

$$\varepsilon = g\alpha\beta\Phi \,, \quad F_c = c_p\, \varrho\beta\chi\Phi \,, \tag{24}$$

where, see AMLT Eq. (A14),

$$\Phi = \tfrac{1}{2} \textstyle\sum^{-1} [(1+\sum)^{1/2} - 1]^3 \,, \quad \sum = U^{-2}(x)\,(\nabla - \nabla_{ad}) \,. \tag{25}$$

The net result is that ν_t resulting from (23) is a rather complicated function of the two variables x and S,

$$\nu_t = \nu_t(x, S) \,. \tag{26}$$

An analogous expression holds for χ_t. Inserting (26) into (22) and then into (15), one can solve for n_s which can then be used in (14). The resulting $x = x(S)$ relation is given by complicated algebraic expressions of no direct interest. We have numerically solved these expressions and then fitted the $x = x(S)$ values. The final result is (with a Kolmogoroff constant $\mathrm{Ko} = 1.65$, $x_0 = 1.21$)

$$x = 1 + S(32 + 4.8\, S^{3/4} + x_0^{-1}\, S^{9/10})^{-1} \tag{27}$$

to be contrasted with the weak turbulence case given by Eq. (17a). As before, substituting x from (27) into (13) and recalling that

$$S = \alpha^4 S_p \tag{28}$$

we obtain the final expression for α in terms of S_p. Numerical results are presented in Table 1. As one can notice, the vastly different results between the weak and strong turbulence cases are a clear indication of the different regimes of validity of the two models for x, i.e., near the transition to turbulence and a fully developed turbulent regime. For large values of S_p, the strong turbulence results can be represented by the simple expression, $\alpha_* = (\alpha_0\, x_0^{1/2})^{5/4}$

$$\alpha = \alpha_*\, S_p^{1/16} \tag{29}$$

to be contrasted with the weak case, Eq. (21).

The values of Table 1 and the representation given by Eq. (29) represent our main result expressing α in terms of local thermodynamic variables.

Finally, one would expect the degree of anisotropy x to be far larger in the unrenormalized, weak turbulence case than in the strong turbulence regime since the presence of strong non-linear interactions is expected to break up incipient, largely anisotropic, eddies. This results is indeed borne out by the above formulae since

$$\text{Weak turbulence:} \quad x \approx S^{1/4} \,,$$
$$\text{Strong turbulence:} \quad x \approx S^{1/10} \,, \tag{30}$$

which shows that the degree of anisotropy is considerably less in the strong turbulence regime than in the weak case.

4. Conclusions

In a previous paper, we constructed a mixing length theory valid for an arbitrary degree of anisotropy. In this paper we show that basic relation between $\nabla - \nabla_{ad}$, $\nabla - \nabla_r$ and U (i.e., the local variables) has retained the same formal expression as in the ordinary isotropic MLT, i.e., Eq. (5), *provided* the mixing length l is interpreted as $l(x)$, the two being related by Eq. (8). This reinterpretation has two effects: first, one can formally use the MLT formalism provided that the α so derived is identified with $\alpha(x)$, a function related to α by Eq. (11). Once a value of $\alpha(x)$ is obtained from stellar structure calculations, Eq. (11) can be used to construct a physical α which is now a local variable. The results for the cases of weak and strong turbulence are given by Eqs. (20) and (29) respectively.

Alternatively, one can say that the MLT *correctly* computes a global quantity $\alpha(x)$ representing an idealized isotropic turbulent regime, but then *incorrectly* identifies it with the physical α which is expected to depend on the local variables. The basic assumption underlying the MLT, i.e., isotropy in the eddies spectrum, clearly precluded the differentiation between α and $\alpha(x)$.

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